

ON A GENERALIZATION OF CRITERIA A AND D FOR CONGRUENCE OF TRIANGLES

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ABSTRACT. The conditions determining that two triangles are congruent play a basic role in planimetry. By comparing not congruent triangles with respect to given sets of corresponding elements it is important to discover if they have any common geometric properties characterizing them. The present paper is devoted to an answer of this question. We give a generalization of congruence criteria A and D for triangles and apply it to prove some selected geometric problems.

1. INTRODUCTION

There are six essential elements of every triangle - three angles and three sides. The method of constructing a triangle varies according to the facts which are known about its sides and angles.

It is important to know what is the minimum knowledge about the sides and angles which is necessary to construct a particular triangle.

Clearly all triangles constructed in the same way with the same data must be identically equal, i. e. they must be of exactly the same size and shape and their areas must be the same.

Triangles which are equal in all respects are called *congruent triangles*.

The four sets of minimal conditions for two triangles to be congruent are set out in the following geometric criteria.

Criterion A. *Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.*

Criterion B. *Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and a side of the other.*

Criterion C. *Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other.*

Criterion D. *Two triangles are congruent if two sides and the angle opposite the greater side of one triangle are respectively equal to two sides and the angle opposite the greater side of the other.*

It should be noted that in *criteria A* and *D* the sets of corresponding equal elements are two sides and an angle.

In fact the angle given may be any one of the three angles of the triangle. The problem to “Construct a triangle with two of its sides a and b , $a < b$, and angle α opposite the smaller side” has not a unique solution. There can be two triangles each of them satisfying the given conditions.

In the present paper we compare not congruent triangles with respect to given sets of corresponding elements and answer the question what are the geometric properties characterizing such couples of triangles.

2. THEORETICAL BASIS OF THE PROPOSED METHOD FOR COMPARING TRIANGLES

In $\triangle ABC$ and $\triangle A_1B_1C_1$ it is convenient to use the notations $AB = c$, $BC = a$, $CA = b$; $A_1B_1 = c_1$, $B_1C_1 = a_1$, $C_1A_1 = b_1$. Let θ and θ_1 be two corresponding angles of these triangles.

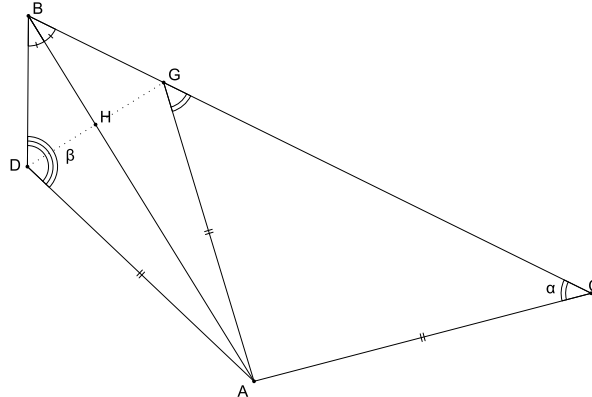
If for $\triangle ABC$ and $\triangle A_1B_1C_1$ the relations $a = a_1$, $b = b_1$ and $\theta = \theta_1$ between the corresponding elements hold, we consider the following cases.

- The angle θ is included between the sides a and b , i. e. $\theta = \angle ACB$, and respectively $\theta_1 = \angle A_1C_1B_1$. In this case the triangles are congruent in view of *Criterion A*.
- Let $a = b$ and correspondingly $a_1 = b_1$, i. e. $\triangle ABC$ and $\triangle A_1B_1C_1$ are isosceles. Since $\theta = \theta_1$, the triangles are congruent as a consequence of *Criterion A*.
- Let $a > b$, correspondingly $a_1 > b_1$, and the angle θ is opposite the greater side a . In this case the triangles are congruent in view of *Criterion D*.
- Let $a > b$, correspondingly $a_1 > b_1$, and the angle θ is opposite the smaller side b . In this case the triangles are either congruent or not.
 - If the triangles are congruent, then the angles opposite the greater sides are necessarily equal. It could happen that the sum of the equal angles opposite the greater sides is two right angles. If so, the triangles are right-angled.
 - If the triangles are not congruent, then we show that the sum of the angles opposite the greater sides is always two right angles.

We prove the following

Lemma 2.1. *Let $\triangle ABC$ and $\triangle ABD$ be not congruent triangles having AB as a common side. Let also $AC = AD$. If $\angle ABC = \angle ABD$, then $\angle ACB + \angle ADB = 180^\circ$.*

Proof. Since $\triangle ABC$ and $\triangle ABD$ are not congruent, then $AC < AB$ (and hence $AD < AB$). Let us denote $\angle ACB = \alpha$ and $\angle ADB = \beta$. There are two possibilities for the location



of $\triangle ABC$ and $\triangle ABD$ with respect to the straight line AB .

(i) *The points C and D lie on opposite sides of AB .*

The symmetry with respect to the straight line AB transforms $\triangle ABD$ into its congruent $\triangle ABG$ which lies on one and the same side of the axis of symmetry AB like $\triangle ABC$ (fig. 1). Since $\triangle ABC \not\cong \triangle ABD$, then $\triangle ABC \not\cong \triangle ABG$. The condition $\angle ABC = \angle ABD$ states that the straight line AB is the bisector of $\angle DBC$. From the symmetry with respect to AB it follows that $G \in BC$ and $BG \neq BC$. Let, for instance, G/BC . (The case C/BG is analogical.) It is clear that if the conditions of Lemma 2.1 are fulfilled for $\triangle ABC$ and $\triangle ABD$, then they are also valid for $\triangle ABC$ and $\triangle ABG$ and vice versa.

Let us consider $\triangle ABC$ and $\triangle ABG$. The side AB and $\angle ABC$ are common for both triangles. In view of the symmetry with respect to AB and $AC = AD$, we get $AD = AG = AC$. Hence, $\triangle ACG$ is isosceles and $\angle ACG = \alpha = \angle AGC$. The angles $\angle AGC$ and $\angle AGB = \angle ADB = \beta$ are adjacent and hence $\angle AGC + \angle AGB = \angle ACB + \angle ADB = \alpha + \beta = 180^\circ$.

Remark 2.2. The quadrilateral $ACBD$ can be inscribed in a circle.

(ii) *The points C and D lie on one and the same side of AB .*

We consider $\triangle ABC$ and $\triangle ABG$ (in this case $D \equiv G$). The proof of the statement is as in (i). \square

Remark 2.3. If $\triangle ABC$ and $\triangle A_1B_1C_1$ are not congruent, the relations $AB = A_1B_1$, $AC = A_1C_1$ and $\angle ABC = \angle A_1B_1C_1$ between their corresponding elements are fulfilled and they have no common side, then we can choose a suitable congruence and transform $\triangle A_1B_1C_1$ into its congruent $\triangle ABD$ so that both triangles satisfy the conditions of Lemma 2.1.

Based on the above arguments we can formulate a theorem, which is a generalization of criteria A and D for congruence of triangles (see also [6], p. 12).

The denotations $AB = c$, $BC = a$, $CA = b$; $A_1B_1 = c_1$, $B_1C_1 = a_1$, $C_1A_1 = b_1$ are usually used in $\triangle ABC$ and $\triangle A_1B_1C_1$ respectively.

Theorem 2.4. *Let θ and θ_1 be two corresponding angles of $\triangle ABC$ and $\triangle A_1B_1C_1$. If $a = a_1$, $b = b_1$ and $\theta = \theta_1$, then $\triangle ABC$ and $\triangle A_1B_1C_1$ are either congruent, or not congruent but the sum of the other two angles, not included between the given sides, is two right angles.*

Lemma 2.1 and Theorem 2.4 can be used as alternative methods of comparing different triangles.

3. APPLICATION OF THEOREM 2.4 TO TWO GEOMETRIC PROBLEMS

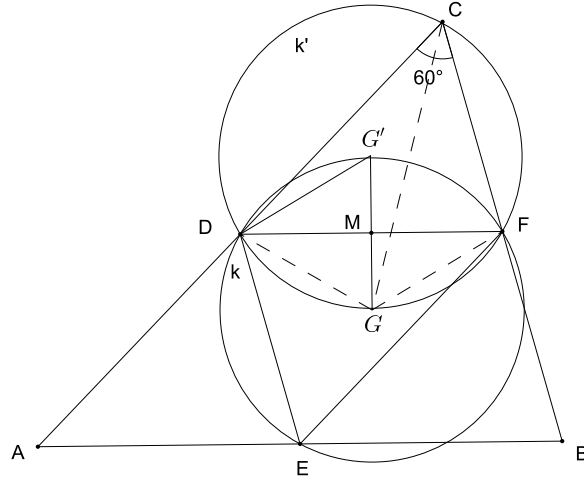
The solutions of next selected problems are based on Theorem 2.4.

Problem 3.1. ([4], problems 4.20 and 4.23; [5]) *Let the middle points of the sides BC , CA and AB of $\triangle ABC$ be F , D , and E respectively. If the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$, then $\triangle ABC$ is either isosceles ($CA = CB$), or not isosceles but $\angle ACB = 60^\circ$.*

Proof. It is given that the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$ (fig. 2).

Since $\triangle CGD$ and $\triangle CGF$ have a common side CG , equal corresponding angles $\angle DCG = \angle FCG$ and equal corresponding sides $DG = FG$ (as radii of k), the conditions of Theorem 2.4 are satisfied. Then $\triangle CGD$ and $\triangle CGF$ are either congruent, or not congruent.

(i) If $\triangle CGD$ and $\triangle CGF$ are congruent, then $CD = CF$ and hence $CA = CB$, i. e. $\triangle ABC$ is isosceles.



Remark 3.2. There are two possibilities for $\angle ACB$:

- either $\angle ACB = 60^\circ$ and $\triangle ABC$ is equilateral,
- or $\angle ACB \neq 60^\circ$ and $\triangle ABC$ is isosceles.

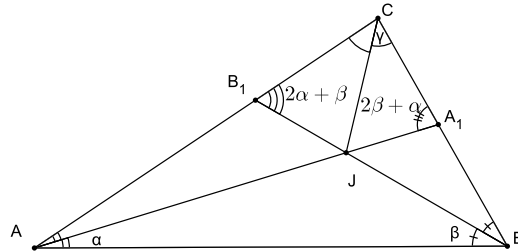
(ii) If $\triangle CGD$ and $\triangle CGF$ are not congruent, then in view of Lemma 2.1 $\angle CDG + \angle CFG = 180^\circ$ and the quadrilateral $CDGF$ can be inscribed in a circle k' (fig. 2).

It is easy to be seen that $\triangle EFD \cong \triangle CDF$ and the circumscribing circles k and k' have equal radii. The circles k and k' are symmetrically located with respect to their common chord FD . Since the center G of k lies on k' , then the center G' of k' lies on k . Hence, $\triangle DGG' \cong \triangle FGG'$, both triangles are equilateral, $\angle DGF = 120^\circ$ and $\angle ACB = 60^\circ$. \square

Problem 3.3. ([3], Problem 8; [4], Problem 4.12) Let in $\triangle ABC$ the straight lines AA_1 , $A_1 \in BC$, and BB_1 , $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$ respectively. Let also $AA_1 \cap BB_1 = J$. If $JA_1 = JB_1$ then $\triangle ABC$ is either isosceles ($CA = CB$), or not isosceles but $\angle ACB = 60^\circ$.

Proof. We use the denotations $\angle BAC = 2\alpha$, $\angle ABC = 2\beta$, $\angle ACB = 2\gamma$.

Since J is the cut point of the angle bisectors AA_1 and BB_1 of $\triangle ABC$, then the straight line CJ is the bisector of $\angle ACB$ and $\alpha + \beta + \gamma = 90^\circ$ (fig. 3).



Since $\angle CB_1J$ is an exterior angle of $\triangle ABB_1$, then $\angle CB_1J = 2\alpha + \beta$. Since $\angle CA_1J$ is an exterior angle of $\triangle ABA_1$, then $\angle CA_1J = 2\beta + \alpha$.

Let us compare $\triangle CA_1J$ and $\triangle CB_1J$. They have a common side CJ , corresponding equal sides $JA_1 = JB_1$ and angles $\angle A_1CJ = \angle B_1CJ$.

The conditions of Theorem 2.4 are satisfied. Then $\triangle CA_1J$ and $\triangle CB_1J$ are either congruent, or not.

(i) If $\triangle CA_1J$ and $\triangle CB_1J$ are congruent, then their corresponding elements are equal, in particular

$$\angle CB_1J = \angle CA_1J \Leftrightarrow 2\alpha + \beta = 2\beta + \alpha \Leftrightarrow \alpha = \beta.$$

Hence, $\triangle ABC$ is isosceles with $CA = CB$.

Remark 3.4. There are two possibilities for $\angle ACB$:

- $\angle ACB = 60^\circ$ and $\triangle ABC$ is equilateral;
- $\angle ACB \neq 60^\circ$ and $\triangle ABC$ is isosceles.

(ii) If $\triangle CA_1J$ and $\triangle CB_1J$ are not congruent, then with respect to Lemma 2.1

$$\angle CB_1J + \angle CA_1J = 180^\circ \Leftrightarrow (2\alpha + \beta) + (2\beta + \alpha) = 180^\circ \Leftrightarrow \alpha + \beta = 60^\circ.$$

Hence, $\angle ACB = 60^\circ$. □

4. GROUPS OF PROBLEMS

In this section we illustrate the composing technology of new problems as an interpretation of specific logical models.

Our aim is the *basic problem* in each of the groups under consideration to be with (exclusive or not exclusive) disjunction as a logical structure in the conclusion and its proof to be based on Lemma 2.1 or on Theorem 2.4.

4.1. Problems of group I. Suitable logical models for formulation of *equivalent* problems and *generating* problems of a given problem are described in detail in [3, 4].

The basic statements we need in this group of problems are:

$t := \{ \text{A square with center } O \text{ is inscribed in a } \triangle ABC \text{ in the following way: the vertexes of the square lie on the sides of the triangle, in addition two of them lie on the side } AB. \}$

$p := \{ \angle ACB = 90^\circ \}$

$q := \{ CA = CB \}$

$r := \{ \angle ACO = \angle BCO \}$

We describe the logical scheme for the composition of the Basic problem 4.4, which has not exclusive disjunction as a logical structure in the conclusion:

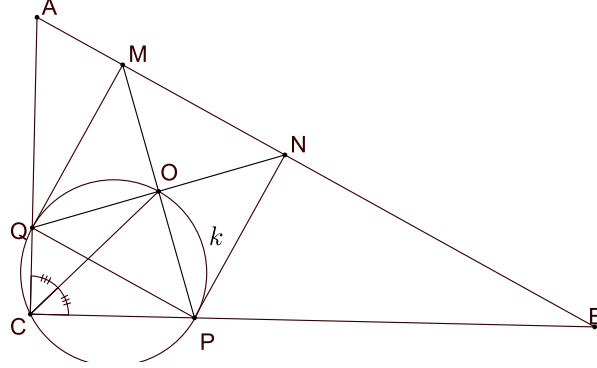
- First we formulate and prove the *generating* problems - Problem 4.1 with logical structure $t \wedge p \rightarrow r$ and Problem 4.3 with logical structure $t \wedge q \rightarrow r$.
- To generate problems with logical structure $(*) \quad t \wedge (p \vee q) \rightarrow r$ we use the logical equivalence

$$(t \wedge p \rightarrow r) \wedge (t \wedge q \rightarrow r) \Leftrightarrow t \wedge (p \vee q) \rightarrow r.$$

- Finally, the formulated *inverse* problem - Basic problem 4.4 - to the problem with structure $(*)$ has the logical structure $t \wedge r \rightarrow p \vee q$.

Problem 4.1. In $\triangle ABC$ is inscribed a square with center O in the following way: the vertexes of the square lie on the sides of the triangle, in addition two of them lie on the side AB . Prove that if $\angle ACB = 90^\circ$, then $\angle ACO = \angle BCO$.

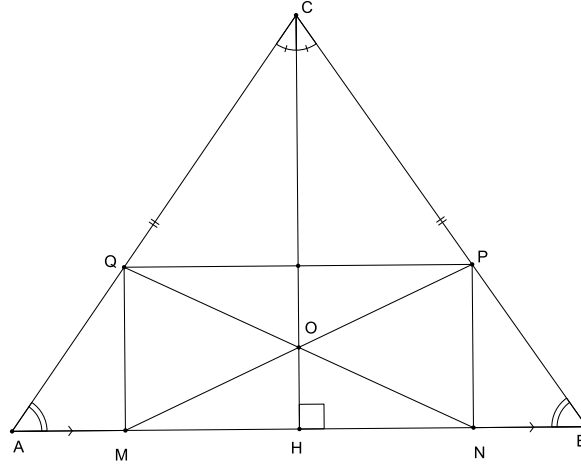
Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (fig. 4). Since the diagonals of a square are equal, intersect at



right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\angle POQ = 90^\circ$. The quadrilateral $OPCQ$ can be inscribed in a circle k with diameter PQ . To the equal chords OQ and OP of k correspond equal angles, i. e. $\angle ACO = \angle BCO$. \square

Problem 4.2. In $\triangle ABC$ is inscribed a rectangle with center O in the following way: the vertexes of the rectangle lie on the sides of the triangle, in addition two of them lie on the side AB . Prove that if $CA = CB$, then $\angle ACO = \angle BCO$.

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ rectangle (fig. 5). Since the diagonals of a rectangle are equal and



bisect each other, then $OM = ON = OP = OQ$.

Let $CH \perp AB$, $H \in AB$. Provided that $\triangle ABC$ is isosceles with $CA = CB$, the point H is the middle point of AB and the straight line CH is the bisector of $\angle ACB$.

Because $MQ \parallel NP$, $NP \parallel CH$ and $MQ = NP$, it follows that $\triangle AMQ \cong \triangle BNP$ (by *Criterion B*) and $AM = BN$. Hence, H is also the middle point of MN . Since $\triangle MON$ is isosceles, then its median OH is also an altitude, i. e. $OH \perp MN$. This means that $O \in CH$ and $\angle ACO = \angle BCO$. \square

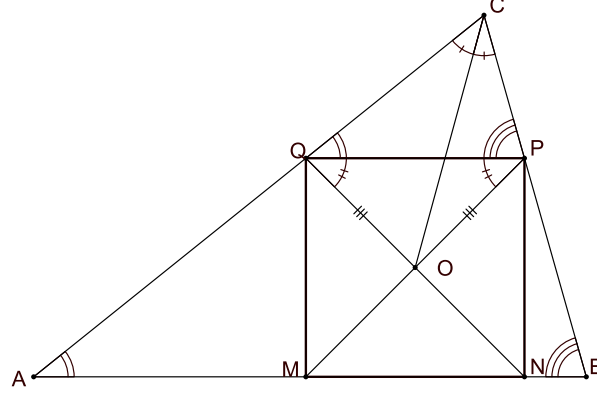
A special case of Problem 4.2 is Problem 4.3 with a logical structure $t \wedge q \rightarrow r$.

Problem 4.3. In $\triangle ABC$ is inscribed a square with center O in the following way: the vertexes of the square lie on the sides of the triangle, in addition two of them lie on the side AB . Prove that if $CA = CB$, then $\angle ACO = \angle BCO$.

Now we formulate and prove the *Basic problem* in this group.

Basic problem 4.4. *In $\triangle ABC$ is inscribed a square with center O in the following way: the vertexes of the square lie on the sides of the triangle, in addition two of them lie on the side AB . Prove that if $\angle ACO = \angle BCO$, then $CA = CB$ or $\angle ACB = 90^\circ$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (fig. 6). Since the diagonals of any square are equal,



intersect at right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\angle OPQ = \angle OQP = 45^\circ$.

We compare $\triangle CQO$ and $\triangle CPO$. They have a common side CO , respectively equal sides $OQ = OP$ and angles $\angle QCO = \angle PCO$. We compute that $\angle CQO = \angle CAB + 45^\circ$ and $\angle CPO = \angle CBA + 45^\circ$ as exterior angles of $\triangle QAN$ and $\triangle PBM$ respectively.

In view of Theorem 2.4 $\triangle CQO$ and $\triangle CPO$ are either congruent or not.

- (i) If $\triangle CQO$ and $\triangle CPO$ are congruent, then $\angle CQO = \angle CPO$ and hence $\angle CAB = \angle CBA$, i. e. $CA = CB$ and $\triangle ABC$ is isosceles.

In this case $\angle ACB$ is either a right angle and $\triangle ABC$ is isosceles right-angled, or not a right angle and $\triangle ABC$ is only isosceles.

- (ii) If $\triangle CQO$ and $\triangle CPO$ are not congruent then, in view of Lemma 2.1, $\angle CQO + \angle CPO = 180^\circ$ and hence $\angle CAB + \angle CBA = 90^\circ$, i. e. $\triangle ABC$ is right-angled with $\angle ACB = 90^\circ$.

□

Remark 4.5. A logically incorrect version of the Basic problem 4.4 is Problem 1.54 in [1].

We reformulate Problem 4.4 by keeping the condition of homogeneity of the conclusion.

Problem 4.6. *In $\triangle ABC$ is inscribed a square with center O in the following way: the vertexes of the square lie on the sides of the triangle, in addition two of them lie on the side AB . Prove that if $\angle ACO = \angle BCO$, then $\triangle ABC$ is either isosceles with $CA = CB$, or not isosceles but right-angled with $\angle ACB = 90^\circ$.*

4.2. Problems of group II. By formulating appropriate statements and giving suitable logical models we get two *generating* problems that are necessary for the construction of the Basic problem 4.9.

The basic statements we need are:

$t := \{ \text{In } \triangle ABC \text{ the straight lines } AA_1, A_1 \in BC, \text{ and } BB_1, B_1 \in AC, \text{ are the bisectors of } \angle CAB \text{ and } \angle CBA \text{ respectively.} \}$

$p := \{ \angle ACB = 60^0 \}$

$q := \{ \angle CAB = 120^0 \}$

$r := \{ \angle BB_1A_1 = 30^0 \}$

Since the sum of the angles of any triangle is equal to two right angles, the statements p and q are mutually exclusive. Hence, if p is true, so is $\neg q$ and vice versa.

We describe the logical scheme for the composition of the Basic problem 4.9, which has exclusive disjunction as a logical structure in the conclusion:

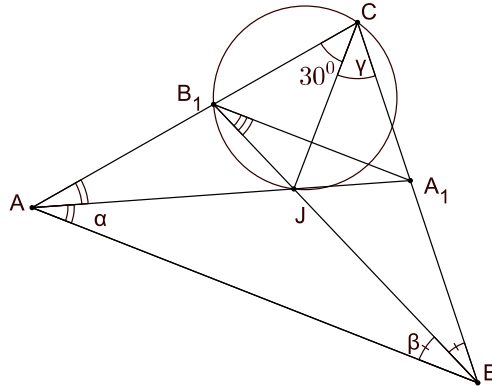
- First we formulate and prove the *generating* problems - Problem 4.7 with logical structure $t \wedge p \rightarrow r$ and Problem 4.8 with logical structure $t \wedge q \rightarrow r$.
- Since the statements p and q are mutually exclusive then the equivalences $p \wedge \neg q \Leftrightarrow p$ and $\neg p \wedge q \Leftrightarrow q$ are true. As a consequence of these facts problems with logical structures $t \wedge p \rightarrow r$ and $t \wedge (p \wedge \neg q) \rightarrow r$ are equivalent. So the problems with logical structures $t \wedge q \rightarrow r$ and $t \wedge (q \wedge \neg p) \rightarrow r$.

To generate problems with logical structure $(**) \quad t \wedge (p \vee q) \rightarrow r$ we use the logical equivalence

$$(t \wedge (p \wedge \neg q) \rightarrow r) \wedge (t \wedge (\neg p \wedge q) \rightarrow r) \quad \Leftrightarrow \quad t \wedge (p \vee q) \rightarrow r.$$

- Finally, the formulated *inverse* problem - the Basic problem 4.9 - to the problem with structure $(**)$ has the logical structure $t \wedge r \rightarrow p \vee q$.

Problem 4.7. *Let in $\triangle ABC$ the straight lines $AA_1, A_1 \in BC$, and $BB_1, B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$ respectively. Prove that if $\angle ACB = 60^0$, then $\angle BB_1A_1 = 30^0$.*



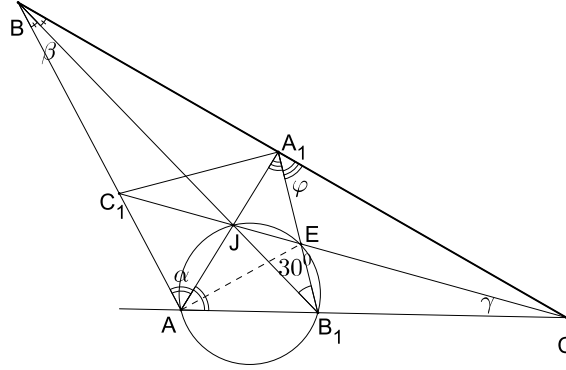
Proof. Let $\angle BAA_1 = \angle CAA_1 = \alpha$, $\angle ABB_1 = \angle CBB_1 = \beta$, $J = AA_1 \cap BB_1$.

Since J is the cut point of the angle bisectors of $\triangle ABC$, then $\angle JCA = \angle JCB = \gamma = 30^0$ (fig. 7).

Because $\alpha + \beta + \gamma = 90^\circ$ it follows that $\angle AJB = 120^\circ$. Hence, the quadrilateral CA_1JB_1 can be inscribed in a circle. Then $\angle JA_1B_1 = \angle JCB_1 = 30^\circ$ and $\angle JB_1A_1 = \angle JCA_1 = 30^\circ$ as angles in the same segment of this circle. \square

Problem 4.8. *Let in $\triangle ABC$ the straight lines AA_1 , $A_1 \in BC$, and BB_1 , $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$ respectively. Prove that if $\angle BAC = 120^\circ$, then $\angle BB_1A_1 = 30^\circ$.*

Proof. Let $J = AA_1 \cap BB_1$, $E = A_1B_1 \cap CJ$, $C_1 = CJ \cap AB$. Since $\angle BAC = 120^\circ$, then its adjacent angles have a measure of 60° . It is easy to be seen that the point B_1 is equidistant from the straight lines BA , BC , AA_1 and that the straight line A_1B_1 is the bisector of $\angle CA_1A$ (fig. 8). The proof that the straight line A_1C_1 is the bisector of $\angle BA_1A$



is analogical. It follows that $\angle B_1A_1C_1$ is a right angle (the bisectors of any two adjacent angles are perpendicular to each other) (see also [2], p. 194, Problem 156).

As a consequence we get that E is the intersection point of the angle bisectors CJ and A_1B_1 of $\triangle AA_1C$ and hence $\angle JAE = \angle EAB_1 = 30^\circ$.

Let $\varphi = \angle CA_1B_1 = \angle B_1A_1A$ and $\gamma = \angle C_1CA = \angle C_1CB$. Then $\angle A_1B_1C = 60^\circ + \varphi$ as an exterior angle of $\triangle A_1B_1A$, the sum of the angles of $\triangle AA_1C$ is $60^\circ + 2\varphi + 2\gamma = 180^\circ$, i. e. $\varphi + \gamma = 60^\circ$ and hence $\angle JEB_1 = 120^\circ$.

Thus, the quadrilateral $AJEB_1$ can be inscribed in a circle. We conclude that $\angle JAE = \angle JB_1E = 30^\circ$ as angles in the same segment of this circle. Hence, $\angle BB_1A_1 = 30^\circ$. \square

Now we formulate and prove the *Basic problem* in this group.

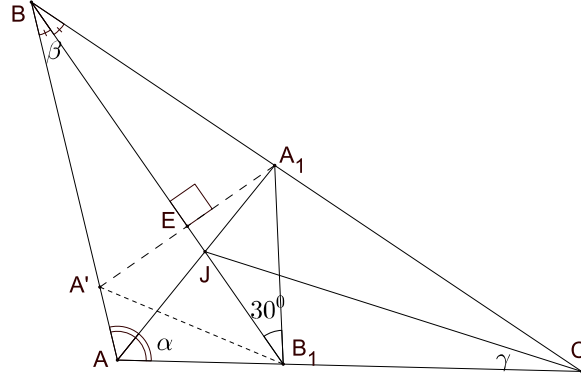
Basic problem 4.9. *Let in $\triangle ABC$ the straight lines AA_1 , $A_1 \in BC$, and BB_1 , $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$ respectively. Prove that if $\angle BB_1A_1 = 30^\circ$, then either $\angle ACB = 60^\circ$ or $\angle BAC = 120^\circ$.*

Proof. Let us denote $\angle BAA_1 = \angle CAA_1 = \alpha$, $\angle ABB_1 = \angle CBB_1 = \beta$, $AA_1 \cap BB_1 = J$.

Since J is the cut point of the angle bisectors of $\triangle ABC$, then the straight line CJ is the bisector of $\angle ACB$. Denoting $\gamma = \angle JCA = \angle JCB$ we get $\alpha + \beta + \gamma = 90^\circ$ (fig. 9).

Let the point A' be orthogonally symmetric to the point A_1 with respect to the axis BB_1 . It follows that $A' \neq A$. (If $A' \equiv A$ then $\triangle ABC$ does not exist.) The straight line BB_1 is the bisector of $\angle ABC$ and consequently $A' \in AB$ and $B_1A_1 = B_1A'$. On the other hand, $\angle BB_1A_1 = 30^\circ$ and hence $\triangle A_1B_1A'$ is equilateral.

We compute $\angle AA'B_1 = 30^\circ + \beta$ (as an exterior angle of $\triangle A'BB_1$), $\angle AA'A_1 = 90^\circ + \beta$ (as an exterior angle of $\triangle A'BE$), $\angle AB_1A' = 60^\circ + \gamma - \alpha$ and $\angle AB_1A_1 = 120^\circ + \gamma - \alpha$.



Let us compare $\triangle AA_1B_1$ and $\triangle AA_1A'$. They have a common side AA_1 and corresponding equal sides $A_1B_1 = A_1A'$ and angles $\angle B_1AA_1 = \angle A'AA_1 = \alpha$. In view of Theorem 2.4 we have the possibilities:

- (i) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are congruent. Then $\angle AB_1A_1 = \angle AA'A_1$, i. e. $120^\circ + \gamma - \alpha = 90^\circ + \beta$. Hence, $2\gamma = \angle ACB = 60^\circ$.
- (ii) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are not congruent. By Lemma 2.1 it follows that $\angle AB_1A_1 + \angle AA'A_1 = 180^\circ$, i. e. $(120^\circ + \gamma - \alpha) + (90^\circ + \beta) = 180^\circ$. Hence, $2\alpha = \angle BAC = 120^\circ$.

□

Remark 4.10. An alternate version of Problem 4.9 is Problem 6, p. 12, in [6].

In order to formulate a special type equivalent problem to this Basic problem we prove

Proposition 4.11. *If the statements p and q are mutually exclusive then the following equivalences are true*

$$(\neg(p \vee q)) \Leftrightarrow (p \vee \neg q) \wedge (\neg p \vee q) \Leftrightarrow \neg p \wedge \neg q.$$

Proof.

$$\begin{aligned} (\neg(p \vee q)) &\Leftrightarrow \neg((p \wedge \neg q) \vee (\neg p \wedge q)) \\ &\Leftrightarrow (p \vee \neg q) \wedge (\neg p \vee q) \Leftrightarrow p \wedge (\neg p \vee q) \vee \neg q \wedge (\neg p \vee q) \\ &\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q) \vee (\neg q \wedge \neg p) \vee (q \wedge \neg q) \Leftrightarrow \neg p \wedge \neg q. \end{aligned}$$

□

Because of this Proposition problems with logical structures $t \wedge (\neg(p \vee q)) \rightarrow \neg r$ and $t \wedge (\neg p \wedge \neg q) \rightarrow \neg r$ are equivalent.

The following problem is equivalent to the Basic problem 4.9.

Problem 4.12. *Let in $\triangle ABC$ the straight lines AA_1 , $A_1 \in BC$, and BB_1 , $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$ respectively. Prove that if $\angle ACB \neq 60^\circ$ and $\angle CAB \neq 120^\circ$ then $\angle BB_1A_1 \neq 30^\circ$.*

Proof. Assuming the truth of the contrary statement, i. e. $\angle BB_1A_1 = 30^\circ$, the solution of this problem leads to the solution of the Basic problem 4.9. □

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